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Abstract

The use of tasks to promote mathematical reasoning (MR) in teaching practice is essential to meet curricular goals. However, that practice is often a huge challenge for teachers, and particularly for prospective teachers and thus it is essential to highlight it as a goal for initial teacher education. This study focuses on preservice mathematics teachers' (PTs) knowledge about the potential of mathematical tasks to promote students' MR, in a teacher education course. Results show that PTs were able to justify their option for a mathematical task with potential to promote students' MR, and through its implementation in one 8th grade classroom they have deepen their knowledge and gave greater meaning to task design principles and acknowledging their students' knowledge. Thus, the activity of selecting and adapting a task, although less demanding than the design of a new task, can still provide PT with important reflection and knowledge about its potential to promote students' MR. The study stresses the relevance for initial teacher education of considering four domains associated with the recognition of the potential of tasks to promote MR.

Introduction

Mathematical reasoning (MR) has received increasing attention in curriculum documents as an essential ability to be developed by all students to acquire and communicate mathematical knowledge and to learn with understanding (Jeannotte & Kieran, 2017; NCTM, 2000; Stylianides & Stylianides, 2007). However, students' learning highly depends on the experience teachers provide them in mathematics lessons, particularly the nature of the proposed tasks and the way they are enacted in the classroom (Boston & Smith, 2011; Breen & O'Shea 2019; Stein, Grover & Henningsen, 1996). Considering that teaching for promoting students' mathematical reasoning is often a huge challenge for teachers, it is essential to highlight it as a focus of work in initial teacher education (Buchbinder & McCrone, 2020; Ponte & Chapman, 2015).

Preservice teachers (PTs) need to be involved in activities to develop the necessary knowledge to create contexts in which MR assumes greater centrality in students' experiences (Herbert & Bragg, 2020; Ponte & Chapman, 2015; Stylianides, Stylianides, & Shiling-Traina, 2013). That knowledge involves, among other aspects, a deep understanding of MR meaning and processes, and how to propose tasks that promote it in their future practice (Ponte & Chapman, 2015; Rodrigues, Brunheira, & Serrazina, 2021). The analysis, selection, and design of tasks with specific characteristics to promote students' MR, may support the development of PTs' knowledge about that kind of tasks. However, that knowledge is particularly complex and still under researched (Davidson, Herbert, & Bragg, 2019).

This study aims at understanding PTs' knowledge about the potential of mathematical tasks to promote students' MR, as they select and adapt a task and then enact it in one 8^{th} grade classroom, in the context of a teacher education course. With this aim, we formulated the following research questions:

- What aspects are recognized and valued by preservice teachers regarding the potential of a mathematical task that aims to promote students' MR?
- How does the teaching practice based on a mathematical task that aims to promote students' MR extends the PTs' knowledge about the task's potential?

Thus, the present study, may contribute to the scarce research on PTs' knowledge about the potential of tasks that promote MR, by establishing a needed analysis framework that integrates a set of categories to put into evidence their knowledge about the potential of these specific tasks. Although focused on a pair of PTs, this research also contributes to understand the affordances of the teacher education course that has been implemented.

Mathematical Reasoning

Mathematical reasoning is often used in mathematics education with distinct and non-consensual meanings. In this study, following several authors (e.g., Jeannotte & Kieran, 2017; Lannin, Ellis, & Elliot, 2011; Mata-Pereira & Ponte, 2018), we assume MR as a process of making justified inferences from previous information, and that can assume the form of inductive or abductive reasoning to obtain new information, and deductive reasoning to validate the produced inferences. This conceptualization of MR also involves various reasoning processes, which Jeannotte and Kieran (2017) integrate into two categories: the first, related to the search for similarities and differences, includes the processes of identifying patterns, comparing, conjecturing, generalizing and classifying; the second one considers validation processes, including justification and proof. This study focuses on central MR processes as conjecturing, generalizing, justifying and, still, on exemplifying, as it supports other reasoning processes (Jeannotte & Kieran, 2017).

Conjecturing, often based on the identification of patterns after an informal phase of exploration of specific examples using inductive reasoning, leads to statements that are believed to be true, but which requires evidence to validate them (Lannin et al., 2011). The conjecture testing is critical in students' MR and can be developed by testing examples, checking whether the conjecture works for other types of objects, and finding counterexamples or proving it using deductive methods. *Generalization* starts from a specific conclusion or conjecture about an idea, property, or procedure to assert that it is common or valid to a set of objects (Jeannote & Kieran, 2017).

Justification involves creating arguments, explaining why they are true and understanding the role of definitions and counterexamples in this process (Jeannotte & Kieran, 2017), and provides convincing reasons for established conjectures, allowing students to make their reasoning clear and increase their conceptual understanding (NCTM, 2000). Recognizing a justification validity is a critical component of MR (Lannin et al., 2011) and is expected to be supported by mathematical procedures, properties and definitions (Mata-Pereira & Ponte, 2018). The formalization and chain of logical justifications leads to proving and, consequently, to deductive reasoning. Finally, *exemplifying* appears associated with other reasoning processes, involving to find examples to support the identification of similarities and differences, and to perform a validation (Jeannotte & Kieran, 2017).

Tasks to Promote Mathematical Reasoning

Tasks are recognized as a resource "mediator to connect between teaching and learning" (Lee, Lee, & Park, 2019, p. 966), offering students diverse learning opportunities associated with their nature and purposes, such as conceptual or procedural understanding, mathematical reasoning, or problem-solving skills (Thompson, 2012). Mathematics education research that addresses task design highlights the importance of teachers creating or selecting and adapting tasks to meet specific learning objectives (e.g., Liljedahl, Chernoff, & Zazkis, 2007; Thompson, 2012). In fact, the tasks included in textbooks, which teachers use most and rely on to plan their teaching (Kaur & Lam, 2012), often not allow to achieve the important curricular objectives, particularly regarding the MR.

The tasks' characteristics considered as essential to stimulate students' reasoning are the possibility of using multiple representations and solving strategies (Stein et al., 1996), and to contemplate specific MR processes. Research has highlighted processes such as: explore and formulate conjectures and generalizations, requesting explanations and/or justifications of responses (Brodie, 2010; Thompson, 2012); identify counterexamples, and present and evaluate arguments, including mathematical argumentation of colleagues and teacher (Stein et al, 2008; Thompson, 2012); generate examples, evaluate mathematical statements, and use definitions to classify mathematical objects (Breen & O'Shea, 2019). It has also been suggested that tasks should include questions that encourage the formulation of conjectures and generalizations, involving students in purposeful or systematic observation of specific cases and in the search for similarities and differences among objects, as well as from their previous knowledge and other generalizations already known, but changing the conditions of the situation (Jeannotte & Kieran, 2017; Lin et al., 2011).

Preservice Teacher Education and Knowledge to Promote MR

The teachers' and PTs' knowledge necessary to answer the challenges and demands of mathematics education, particularly on MR, is mentioned by several authors. According to Lannin et al. (2011), PTs need to be able to identify and understand MR meaning and how to integrate tasks and learning experiences that promote the students' MR in their classes. For that, Hill, Ball and Schilling (2008) highlight the needed ability to notice students' reasoning, and to select or adapt tasks and use appropriate teaching strategies for a particular group of students, framing their goals with the ones of the class in which they are proposed, and the students' prior knowledge. Thus, to select and modify tasks PTs need to develop two types of knowledge (Ball, Thames, & Phelps, 2008): Subject Matter Knowledge (SMK), for instance about the intended RM processes, and especially Pedagogical Content Knowledge (PCK).

Therefore, it is essential to create opportunities for PTs to select, evaluate, or adapt tasks to propose in the classroom, considering their learning objectives and characteristics that promote them, and thus allowing PTs to identify the new knowledge included in mathematical tasks and to recognize their potential and limitations for students' learning according to curriculum standards (Hill et al. 2008; Lee, Coomes, & Yim, 2019; Son & Kim, 2015). In addition, Lee, Lee and Park (2019) suggest exploring noticing-oriented activities with PTs, including examples of students' work on the tasks, to develop their ability to better modify mathematical tasks, as this process is connected to a progressive understanding of their related mathematical and pedagogical elements as well as of their limitations (Liljedahl et al., 2007). Providing and discussing theoretical frameworks to increase the recognition of specific characteristics of mathematical tasks to achieve outlined objectives can be an essential first step for PTs acquire knowledge to select and modify tasks according to those goals (Dempsey & O'Shea, 2020).

Some recent studies have focused on PTs' conceptions and knowledge to select, modify, and enact mathematical tasks, particularly problems and inquiry tasks, that is high-level cognitive demand tasks (Dempsey & O'Shea, 2020; Kilic et al., 2017; Leavy & Hourigan 2020; Lee, Lee, & Park, 2019; Magiera & Zambak, 2020). To analyse or evaluate PTs' conceptions about those tasks and their ability to recognize the tasks' potential and to elaborate and modify them, these studies identify different aspects PTs need to attend and integrate into the tasks. These comprise including significant contexts for motivating students, the nature of the questions (e.g., exploration) and associated mathematical aspects to be explored by students that are appropriate to their cognitive level and skills (Lee, Lee, & Park, 2019). Within these teacher education programs, PTs have developed their ability to elaborate or modify problems that are curricular framed and to argue about the importance of the tasks' level of challenge being appropriate to students' knowledge and skills and of enabling multiple strategies and solutions, contributing to encourage them to get involved in the activity (Leavy & Hourigan, 2020), as well as to consider teaching strategies, including generating mathematical explanations (Magiera & Zambak, 2020), to encourage students to get involved in the exploration and questioning to solve the task and promote their mathematical understanding (Kilic et al., 2017).

These studies show that, by recognizing these design principles, PTs positively modify the original tasks by fitting these into them (Lee, Lee & Park, 2019). They also value its importance in preparing mathematical tasks and, by complementing it with the assessment of students' learning after its implementation, develop perceptions about the potential of tasks (Kilic et al., 2017). However, in most studies the tasks evaluated or designed by the PTs do not consider the classroom situation, thus research still highlights the need to consider further investigation of how PTs implement cognitively complex mathematical tasks in the classroom (Dempsey & O'Shea, 2020; Leavy & Hourigan, 2020; Oliveira et al., 2021), particularly those targeting students' RM.

Context

Aiming to develop PTs' knowledge to promote students' MR, an instructional unit was carried out with six PTs that were enrolled in the 2nd year of a master's program in the teaching of mathematics for middle and secondary school levels. The eight-sessions instructional unit (2 hours each), taught by the first author and observed by the second, were part of one semester methods course targeting the development of PCK in several mathematical domains. The sessions consisted mainly of workshops where PTs solved instructional tasks focused on aspects such as: MR meaning, tasks' characteristics to promote RM, and teachers' actions (see Table 1). The instructional tasks were explored autonomously by the PTs, individually or in pairs, and collectively discussed. A framework concerning MR (adapted from Jeannotte & Kieran, 2017), and task design theoretical principles to promote it was made available for discussion with the PTs. The addressed principles focus on the MR processes assumed as central, such as generalization and justification (Lin et al., 2011), in addition to general principles that can guide the selection or design of the tasks, such as allowing diverse solving strategies and different representations and encouraging students to reflect on their solutions, under the framework of exploratory teaching approach (Menezes, Canavarro, & Oliveira, 2012).

Table 1. Typology and Focus of the Instructional Unit Sessions

Sessions typology	Focus
Discussion of theoretical texts about MR and its teaching	MR meaning (definition, types, and processes); principles of task design; and teacher's actions to promote students' MR.
Exploration and discussion of instructional tasks, which include mathematical tasks proposed to students, students' work, classroom episodes illustrating students' MR processes and teacher's actions to promote those processes.	Analysis of the tasks' potential to promote MR in algebra and geometry; types and processes of MR revealed by students and their difficulties; teacher's actions that contribute to promote the students' MR.
Oriented to the teaching practice	Sharing and reflection, in two moments of oral presentation with collective discussion, of: objectives and potential of the task developed by the PTs and their intention to promote MR; dynamics of the class in which it was proposed, including teacher actions; student's resolutions considering the MR used; task modification to be considered.

As all PTs were also in the practicum, they were invited to plan and teach one lesson, based on a task they selected and adapted to promote MR in algebra or geometry. Moreover, promoting moments of reflective conversation, involving a discussion among PTs, can provide collective reflections and knowledge development for their future practices (Aparicio Landa, Sosa Moguel, & Cabanas-Sanchez, 2021). So, these tasks were presented and discussed with their colleagues and the teacher educators in one of the sessions. At the end of the instructional unit, the PTs also presented an oral reflection about the taught lessons with these tasks.

Method

The research assumed a qualitative nature (Erickson, 1986), targeting the knowledge about the tasks' potential to promote students' mathematical reasoning evidenced by a pair of PTs, named Júlia and Sandra (fictitious names). These two PTs were selected as they were the only ones of the group that were teaching two different 8^{th} grade classes at the same school. Thus, choosing these pair of PTs as participants could allow us to observe if they considered the students' specificity of each class in the design of a similar task.

Data collection for this study included: (i) participant observation by the teacher educators of the instructional unit sessions; (ii) written documents produced by PTs, including the lesson plan and the task, and a final reflection after the lesson; and (iii) a final interview with the two PTS, conducted by the second author (see Table 2). In the interview, they were asked to report: their understanding of the meaning and importance of MR; the potential or limitations of the task they took into practice, and the adopted teaching approach that contributed to promote students' MR; and the changes they would propose to improve the task, based on the students' difficulties they have identified.

Table 2. Data Collection Sources, Type and Adopted Codes

Sources	Type	Codes
Instructional unit sessions:		
- PTs' oral presentation and discussion of their task's initial version	Video-record	OP1
- PTs' oral reflection and discussion on the taught lesson	Video-record; PPT	OP2; PPT
PTs' lesson plan and task	Written documents	LP
PTs' individual reflection on the taught lesson	Written documents	R
Interview to the PTs	Video-record	I

Data analysis concerns two phases of the PTs' work, based on the framework categories described in Table 3, adapted from the studies referred above since they do not focus on MR, to evidence the PTs' knowledge on the elaboration or adaptation of tasks with specific characteristics to promote MR, in association with the conceptual framework of the MR that was explored in the instructional unit. The first phase respects the initial version of the task selected and adapted by the PTs, and the lesson plan focusing particularly on the characteristics that they considered and recognized as having potential to promote their students' MR, as well as the intended teaching approach. The second phase regards the PTs' post-lesson reflection on the task's potential and the adopted teaching approach, including their interpretation of students' MR and the limitations and affordances that they identify in the original task; and the proposed changes to the task and the teaching approach, including the associated reasons.

Table 3. Potential of the Proposed Task and Teaching Approach to Promote Students' MR

Characterizing elements
General aspects, including: • task nature, its source and type, and its curriculum framework (topic, objectives, and knowledge to be developed) • intentionality of the context (associated with recognized students' prior knowledge and abilities/difficulties, class situation)
General principles of task design involving the use of diverse solving strategies, different representations and encouraging students to reflect on their work
Specific task design principles to promote RM, including the contemplated reason processes
Teaching approach

To guarantee the validity and reliability of the analysis, the two researchers carried out an independent first analysis of the PTs' documents that were afterwards compared. Disagreements or doubts, regarding to the data coding, were discussed and refinements were made to reach a consensus. In the following chapter we present the results of the analysis exemplified with excerpts from the PTs work, organized according with the two mentioned phases.

Results

Phase 1

The task "Sequences" chosen by this pair of PTs for the class that each one taught came out of a search that they did in two books. The choice for a task on sequences was a suggestion from the school cooperating teacher, since he intended PTs to work on a different theme from the one that he was teaching in that period (geometry). The task's statement includes a representation of a sequence consisting of three-dimensional figures (see Figure 1).

When researching possible sequences to be presented to the class, the PTs located this situation that they found not to be totally disconnected from the theme that the students were studying at the time, as it is supported by geometric figures. The task also provided, in their opinion, a familiar context for students, since figures of this type had appeared in a previous class taught by them, having a good receptivity by the students. Sandra explained that: "This figure appears in the sequence of the lesson that we taught about translation, that is why we will use the figure that we used in the task..., to make the bridge... the theme is not the same, but they... will clearly identify as the figure we used in that task" (OP1).

The task

We present the first three terms of a sequence of figures. Each figure represents one construction with cubes which are geometrically equal.

Figure 1 **Figure 2** **Figure 3**

a. How many cubes are necessary to build figure 6? Explain your thinking.
b. Is there any figure in the sequence with 24 cubes? Why?
c. How many cubes are necessary to build figure 10? Explain your thinking.
d. How many cubes will be in figure 64? Justify your answer.
e. Find a general term that allows you to find the number of cubes of any figure of the sequence.
f. Which is the number of the figure that has 64 cubes? How did you think?

Figure 1. The Mathematical Task "Sequences"

The definition of the exact sequence (in 2D or 3D) and the set of questions to be presented in the task, demanded a lot of attention from the PTs, who tried to balance the task's challenge and the accessibility, so that students could develop MR processes. Thus, they decided for the sequence of 3D-figures, to create a challenge for students, namely the need to visualize cubes that are hidden in each figure. They also adapted the original sequence, to be of type $(n + 1)^2$ and instead of n^2, as the original to increase its difficult:

> In fact, we had to adapt the task [from other tasks]. We spent many hours looking and questioning if that was the right task, if we should put it in two dimensions, or in three dimensions… (Júlia, I)
> We put the sequence exactly in the same way, but with 2D, with squares. But that way it would entail a limited [challenge]... I mean, we also want them to evolve mathematically... we want them to learn to reason... (Júlia, OP1)

Additionally, considering that the general expression is a 2nd grade polynomial, which the students are not familiar with, the PTs structured the task through several questions to support the generalization process, before asking for the general expression of the sequence. Júlia explains that: "I imagine that they had not seen as a general rule, *n* plus one, squared. [That is why] we have, to a certain extent, to guide them minimally, without guiding too much" (OP1).

For the PTs, this task was not aimed at learning new mathematical concepts or procedures, but it would allow to appeal to students' prior knowledge, such as the notion of sequence, perfect squares and the use of algebraic symbology, to produce a generalization translated into an unfamiliar algebraic expression, also promoting the students' mathematical reasoning, as expressed in the objectives outlined in their lesson plan: "i) To appeal to students' algebraic thinking; ii) To promote and explore the students' mathematical reasoning; iii) To promote communication and the development of mathematical language, appealing to explanations and justifications of strategies" (LP).

Another aspect that PTs considered in the task's design was the difference they recognized in the mathematical performance and autonomy between the students of the two classes: "In my class, as they present lower levels of achievement and motivation, I chose to tear apart the task a little more, to support the students' activity path and avoid that they become frustrated" (Sandra, R). Thus, observing that Sandra's class usually has more difficulties in carrying out the mathematical tasks proposed, they felt the need to adjust some questions, to support these students. For example, specifically for this class, a table (see Figure 2) was presented to "assist students in organizing the data" (OP2), in question b.

b. Complete the following table from the observation of the sequence of figures

Figure number				
Number of cubes				

Figure 2. Question b) of the Task Adapted for Sandra's Class

Although structured by a set of single answer questions (assuming that the growing sequence follows the pattern indicated by the first terms), the PTs have considered that the task allows a variety of solving strategies, so it does not guide the students in a predefined direction. In the lesson plan, they have anticipated that to determine the figure in position 10 (see Figure 3), students could use two different strategies. PTs show that students may use recurrence between consecutive terms or writing each term of the sequence as a perfect square.

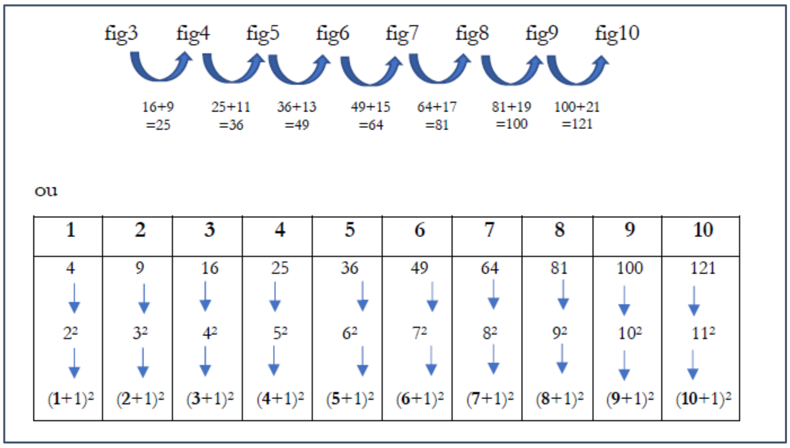

Figure 3. Anticipated Students' Answers for Question c) (LP)

When selecting the task, the PTs seem to have considered that it would allow students to use different representations. For example, in their lesson plan, the PTs raise hypotheses about two ways of conceiving the growing pattern, using different representations. They consider that students may rely upon the figures to grasp the geometric construction, namely realizing "that from Figure 1 to Figure 2, 5 cubes were added and from Figure 2 to Figure 3, 7 cubes were added" (LP) or that they may only focus on the numerical sequence they obtain from there. It is also visible that the PTs recognize the importance of using representations to organize the data (Fig. 3), and that they value students' use of algebraic symbolic and natural language when generalizing, mentioning that "We may expect students to answer in two ways: natural language . . . and $(n+1)^2$" (LP). We also observe that PTs included questions that encourage students to reflect on their resolutions, by asking them to "Explain how you thought" and "Why?", as exemplified in task's questions a) to c) and f).

In the learning objectives outlined for the task that are comprised in the lesson plan, it appears that specific MR objectives have been contemplated, namely "Conjecture about the construction of sequences", "Deduct the general expression [of the term of the numerical sequence associated with the sequence]" and "Justify procedures", which they contend aim at developing "students' mathematical reasoning" (LP). The task structure aims to, through successive steps, support students to reach a generalization of the relationship between term and respective order, that is to find the property that characterizes any term in the sequence and that allows to determine it by knowing its order. The PTs also recognize the role of specific examples in generalizing the relation to all terms in the sequence, mentioning that students "begin the generalization process for the figure 10 and consequently for the general term" (LP).

PTs also included questions that encourage generalizations based on observed similarities and differences between objects. This is evident in the first questions that leads to the identification of a pattern, by asking students to find and register new terms of the sequence. The PTs are attentive to this aspect as they mention: "A question that we thought about to include or not, which we ended up not to include, is asking the students how they saw the figure growing" (Julia, OP1).

The task also addressed the justification, although students are explicitly asked to justify their answer in questions b) and d). Nevertheless, there are other questions that the PTs assume as requests that aim at justification, in which students are asked to explain how they thought. For example, regarding to the first question, PTs mention that students are expected to present calculations or use iconic representations of the terms of the sequence to support their answer:

> [The justification] it can appear in the first [question]. The answer itself can be considered a justification... And the calculation that we expect them to do can be considered a justification, right? (Sandra, OP1)
>
> Normally, they will draw to justify what they do… (Júlia, OP1)

Curiously, in question e) in which it is intended students to obtain the general term of the sequence, which is the main objective of the task, students are not asked to justify or explain their answer. This aspect is not comprised in the lesson plan either. However, PTs referred to justification in association with the conjecturing process, assuming its validation or refutation role. For example, they anticipate that, in an initial phase, when students start exploring the task and try to understand the pattern and determine close terms, they may formulate conjectures that later need to be tested to validate or refute them: "... I think that in question 5 conjectures will arise. Then there will be a refutation or not" (Sandra, OP1).

In what respects the teaching approach to adopt, the PTs express the intention of privileging the exploratory teaching method, organizing the class in three phases: "Task's introduction", "Autonomous work" and "Discussion and Systematization" (LP). In this context, they emphasize that in the lesson planning they considered a set of teacher's actions compatible with this teaching perspective, that have contacted with in the course, namely "anticipation, monitoring, selection, sequencing and establishing mathematical connections" (Sandra, R). Showing coherence with the perspective of allowing space to students' MR when solving the task, and not to conditioning the strategies or representations that they might resort to, PTs state that they intend to support them through questioning, in blocking moments: "Helping students to unblock their reasoning and mathematical thinking through questioning, taking care not to do [their reasoning] by ourselves" (LP). The mentioned principle concerning student's encouragement to reflect on their answers and reasoning processes, is also reflected in the role they intend to assume, as mediators, in the collective discussion of the students' work:

> After [the presentation of] each student, the teacher should question the class about their opinion and favour the exchange of ideas and their justifications by the students, playing here the role of mediator of the discussion. At that time, colleagues will be able to raise their doubts and the student will be asked to explain her/his reasoning. (LP)

Phase 2

The PTs make a positive evaluation of the task's enactment, in the two different versions intended for each class, as they consider that the students had a higher involvement and performance than it was expected by them. Sandra is very pleased with the fact that, notwithstanding her class usually reveals difficulties in learning, in general, students seemed to be at ease in recognizing the structure of the sequence and that some groups were able to generalize algebraically the relationship:

> The students exceeded my expectations. When preparing the lesson, I had predicted that students would reveal more difficulties, namely in identifying that the number of cubes in each figure represented perfect squares, but students were able to do that almost at the beginning of the activity. Moreover, some of the groups obtained the general sequence expression [in symbolic language], which surprised me a lot. (Sandra, R)

Júlia, as well, is surprised with most students' work, namely those who represented the general rule algebraically. She highlights the work of a pair of students (see Figure 4), although presenting some inaccuracies in the algebraic language, it evidences they adequately understood the relation right in the task's first question: "Here, "n + 1 x n + 1", it is obvious that the parentheses are missing, but he does the calculus well... this student has some difficulties in expressing himself..., but the reasoning is all there" (OP2). Júlia adds that the student "realizes that that it is (1 + 1) times (1 + 1) and this will be 4, and then he does here, in fact, what is asked in figure 6, which actually will be 49" (OP2).

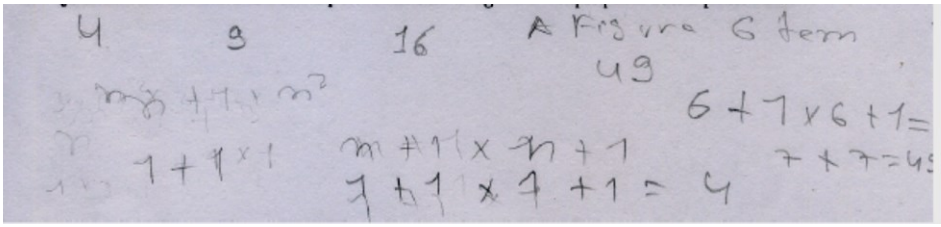

Figure 4. Resolution of Question a) by a Pair of Students (PPT)

Both PTs associate the students' receptiveness regarding the task to the fact that it is not make their work much dependent on previous knowledge, for when that happens, it may block them and make them give up the activity. For instance, Sandra refers that "Students' enthusiasm, especially those who usually are less competent, can be motivated by this type of tasks, as they feel more comfortable in carrying out an activity that does not depend so directly on the previous knowledge they may or may not have, but rather on their involvement during the class" (R). Even so, the PTs identify the task's potential to reinforce the learning of the two classes in the topic of sequences, namely, regarding to the notion of sequence and the use and manipulation of algebraic language: "although the lesson's focus was not on learning the notions intrinsic to sequences, students had opportunity to reinforce the associated language, recalling concepts such as "term" and "order". Learning was also achieved in terms of algebraic expressions manipulation" (Sandra, R).

Regarding the task's potential to promote students' RM, the PTs refer characteristics that they considered in the general principles and recognize their important contribution in this regard. Júlia mentions, among other aspects, that: "the task had different solving strategies; students, based on knowledge acquired in the initial questions, could generalize, reaching the general rule more easily; the questions always asked for justifications of the answers" (R). The PTs also consider that the specific MR design principles they attended in the elaboration of the task and discussed in the instructional unit sessions, fit its objectives. They highlight the opportunities for students' reflection on MR processes provided by the lesson with this task:

> mathematical reasoning processes were privileged, appealing to the explanations and justifications of the strategies used and to generalization based on observation.... The reflection on the reasoning processes used, requesting an explanation on the way they thought... we tried to construct the questions so that they also covered more specific aspects of mathematical reasoning, such as the formulation of generalizations, the justification of answers and the explanation of strategies. (Sandra, R)

> in terms of mathematical reasoning, the lesson was very rich, providing awareness of the importance of justification and explanation of strategies, the need for and importance of generalizing, to facilitate the understanding of what is observed. (Sandra, R)

The PTs analysis of the students' activity, essentially focused on MR processes associated with the task, also allows to understand how they interpret them after their implementation, broadening their vision of the task's potential to promote MR: "each time I look again to the students' work, I am noticing small details that I had not noticed before" (Júlia, OP2). For example, in the case of justification, they make more references about the way students respond to this process, recognizing in what ways they may distinguish from each other, and specifying the inclusion of issues that "also encouraged justifications of different nature (absurd, exhaustion, example)" (Júlia, R), in the task. For example, they refer that students resort to exemplification to show that a certain statement is false, "Refutation and justification by example" (LP), and that students present justifications based on logical coherence: "they conclude that is not possible because there is a figure that has 16 [cubes] and next one that has 25, so there cannot be one [figure] that has 24; therefore it is a justification by logical coherence" (Julia, I). However, no examples of justification for absurd or exhaustion are observed in the lesson plan nor in the students' work that were analysed by the PTs and presented in the instructional unit session. This may indicate that PTs point out this aspect as a generic possibility and that is not specifically related to the potential of this task.

In addition to the characteristics of the proposed mathematical task, the PTs point out the approach adopted as a decisive factor in the emergence of RM processes in this lesson. Júlia refers, particularly, to the contribution of the collective discussion so that the students could present their work and get involved in argumentation processes, namely in the justification of their statements. She mentions that:

> The major factor that contributed to students' mathematical reasoning promotion was to give them time at different moments of the lesson... during the collective discussion, in which I gave students time to explain how they solved that question and justify it, to their colleagues. I also gave space to colleagues to say whether they agreed or not with what was being presented on the board and also asked them to justify why they disagree or not. (Júlia, R)

However, not all teaching strategies adopted by the PTs proved to be equally adequate. Sandra says that the option to have a first discussion right after question e), conditioned some students' work. As such, she felt the need to ask another question so that all students could generalize, without having direct influence of what the fastest colleagues could have done: "what I think that saved the situation was that I then do an extra in which they had to generalize another expression" (Sandra, I).

It should also be noted that, after reflecting on the task and recognizing the importance of intentionally contemplating MR in learning mathematics, the PTs assume that it is possible to promote it through any mathematical task, as long as it can be adjusted to meet specific principles aimed at MR in articulation with the mathematical learning objectives of the topic: "I was able to notice the power of the task as a driver of mathematical reasoning processes and realize that any task can be adapted in such a way that, in parallel with promoting learning in the topic involved, students' mathematical reasoning is stimulated" (Sandra, R). PTs do not identify limitations in the general aspects and principles to achieve the outlined objectives and aimed at the MR that have been considered in the task's design, but they recognize that those were complemented by the actions they developed in the scope of the adopted teaching approach. Still, when reflecting on the students' activity on the task, they point out some changes that they would make in its statement for a possible next application in the classroom. For example, Sandra felt the need to adjust some questions in the task to guide her students' activity as they usually reveal difficulties, but now she questions this decision. She proposes adaptations to the questions, recognizing that, in some cases, the task's challenge level has decreased, which has limited its contribution to develop students' MR:

> I wondered if it would have been necessary to adjust the task by fragmenting and "guiding" the students' strategy. If, on one hand, I think the task was so well received and understood, due to these adjustments, on the other hand, perhaps there was no need to dismember so much the original task. So, I think that the ideal would be to remove question a) ... and amend question c), removing the options. I would keep question b) with the suggestion for using the table, as I believe that this was very important for data organization and for structuring students' thinking. In question h) ... I would replace the number of cubes by a higher value, for example 6084, to prevent students respond through recurrence, and to stimulate the algebraic manipulation of the expression and reverse reasoning. (Sandra, R)

The concern on supporting students' MR is common to the task modifications proposed by Júlia, which are particularly based on a careful choice of specific cases to include in the statement. She considers that should not integrate terms or orders in the questions that make the task too easy for the students, assuming that during its resolution she can bring these values as scaffolds to those who may show difficulties. Thus, she suggests that there is a need "to give a previously thought example of a smaller number in class, so that students feel comfortable and do not give up of the task, than to facilitate the task too much, thus not contributing to the promotion of mathematical reasoning" (Júlia, R). She also argues for the need to adjust the task to the context of the class with which they will be worked, considering that there is no formulation that can be considered suitable for any class: "It is important to note that the task must always be modified according to the class in front of us . . . the changes I am proposing were designed with this class in mind" (Júlia, R).

The changes that PTs proposed go beyond the mathematical task, mentioning that it is necessary to think deeply about the different solving strategies that students can follow. Thus, the anticipation of a variety of solving strategies would be an aspect to improve in the lesson plan, according to the PTs, which allows us to perceive the importance that they attach to this general principle they had pointed out for the task: "in terms of solving strategies, the lesson plan was not very rich, mentioning only one type of resolution for most questions. In practice, we found that the students were more creative than we had anticipated" (Sandra, R).

Both PTs recognize that such limitation in the anticipation of the students' strategies for solving the task, conditioned the effectiveness of their support during the autonomous work, since it hindered their understanding of the students' ideas. For instance, Júlia refers that: "Having only thought in a way to solve the question, I had not envisaged other ways of counting, which meant that I did not immediately understand what the students were explaining" (R). Sandra similarly contends that "Having anticipated other strategies, would have favoured the communication with students and consequently the development of the lesson" (R).

Finally, the PTs also signalled a change of perspective regarding the possibility of proposing tasks with potential to promote students' MR in their classes, which was something that at first seemed involving great complexity. Students' receptiveness seems to have strongly collaborated to that, as stated by Sandra: "before I thought it was a very complicated thing to do... I actually thought: "how am I now going to stimulate students' mathematical reasoning?!" but then I got the perception that it might not be so complicated". She also contends that one of the biggest contributions of these teacher education sessions on RM was "realizing that it is nothing out of this world, that it is possible to do it and also realizing that it is something that students react very well to, that mobilizes them" (I).

Thus, we perceive that these two PTs were globally pleased with the way their classes took place. Still, they were able to identify specific aspects that they could change in future in the lesson planning and in the task, in association with the envisioned teaching approach. We shall discuss these issues in the last section of this paper.

Discussion and Conclusions

This study was carried out in the context of an instructional unit aiming to develop the knowledge of two PTs to promote the students' MR. The work they did in that context, included the selection and adaptation of a mathematical task and its enactment in the classroom, and their final reflection on the taught lessons. This work provided an opportunity to understand their knowledge about the potential of mathematical tasks to promote students' RM.

The results of the study show that the PTs were able to warrant the design of the task from a set of principles addressed in the instructional unit. It should be noted that they sought to balance the task's challenge to promote MR processes as well as its accessibility to students of different levels of achievement, and thus seeking to adapt it in the specificity of the proposed questions, according with what they knew about the students. As in the study by Lee, Lee and Park (2019), the PTs focused not only on the task's mathematical aspects that they intended to be tackled with the class, in this case related to the MR processes, but also attended to their students' knowledge and so that they could successfully carry out the task.

In the rationale for the task's selection and adaptation, the two PTs also evoke general principles for task design, such as allowing diverse solution strategies and the use of different representations, as well as, encouraging students to reflect on their work. These principles entail coherence with the ones that Brodie (2010) and Lin et al. (2011) recognize as relevant to promote MR. Regarding specific principles for task design that aim at MR, the PTs seem to value the incentive to generalization based on the observation of similarities and differences between objects, and particularly questions that ask students to present mathematical justifications for particular cases, a process they also associate with the validation or refutation of a conjecture. However, justification is not explicitly considered by the PTs in the case of the generalization of a general rule, probably because they consider it to have of a higher level of difficulty for these students.

In addition, it is worth noting that PTs make explicit the intention to conduct the class using an exploratory teaching approach, which is consistent with the principles stated for the task design. Thus, PTs seem to value student's autonomy in solving the task, avoiding the teacher from conditioning their strategies, as well as conceiving the moment of collective discussion as an opportunity for students to reflect on the involved MR processes. This perspective is compatible with what Lee, Coomes and Lim (2019) say about teachers who value tasks in which students do not immediately have a strategy to solve them and provide diversified learning experiences in the classroom, but which are less common among preservice teachers due to its level of demand for the teaching practice.

This study shows that the activity of selecting and adapting a task, although less demanding than the design of a new task which may prove to be very difficult for prospective teachers (Leavy & Hourigan, 2020), can still provide them with important reflection and knowledge about its potential to promote students' MR. It also reinforces the recognized relevance of the discussion of theoretical frameworks for the recognition of specific characteristics of mathematical tasks in teacher education (Dempsey & O'Shea, 2020), such as the principles explored in the instructional unit. The possibility of implementing in practice the mathematical task they had developed was a key element in the development of the PTs' knowledge about the potential of the task to promote their students' MR. On the one hand, the feedback they derived from the activity in the two classes allowed them to realize aspects of the task that needed to be modified to ensure that the students were, in fact, involved in MR processes. On the other hand, taking it into practice has increased their conviction about the possibility and relevance of exploring these processes with students, regardless of their level of mathematical proficiency. This classroom experience also allowed them to reassure the adequacy of the principles they had studied in the instructional unit and, mainly, to award them a clearer meaning. Several studies have shown the difficulty for PTs to fully understand theories in the field of mathematics education and apply them with understanding and depth in practice (Lee, Lee & Park, 2019). In fact, the statement of the principles related to the use of different strategies and representations may have been seen in a superficial way, taken by their appearance, and not as effectively different ways of thinking. However, the observation of the diversity of students' ways of thinking gave real meaning to most of these principles, which reinforces the important role of practice in pre-service teacher education.

This research thus reinforces the relevance of considering the exploration of the tasks proposed by preservice teachers in classroom practice, so that they develop a deeper knowledge about the conditions necessary for a more demanding mathematical activity such as reasoning or solving problems, an aspect that has been pointed out as a limitation of other studies (Leavy & Hourigan, 2020). An important component of the teacher's PCK is the ability to consider the students' level of knowledge and abilities and to teach in an adaptive way, estimating the difficulty of a given task, and considering its potential for new learning (Lee, Coomes, & Lim, 2019), which we can report as significant in this study. In fact, what we observed from the PTs in this perspective, although consistent with the results of Kilic et al. (2017), points to the potential of the task with a specific focus on mathematical reasoning, which ascribes relevance to this study.

We can thus conclude that the PTs' recognition of the potential of a task to promote MR is associated with four dimensions (Figure 5) involving knowledge related to: (i) the characteristics of the task, a central aspect, which must follow a set of specific principles of MR (see Table 3); (ii) the nature and processes of MR, as well as their perspective on the relevance of this ability in mathematics learning; (iii) the way in which the task meets students' prior knowledge and the differences between them; and (iv) the teaching approach that they favour, in this case the exploratory teaching based on students' autonomous work in the task and in the collective discussion of their work.

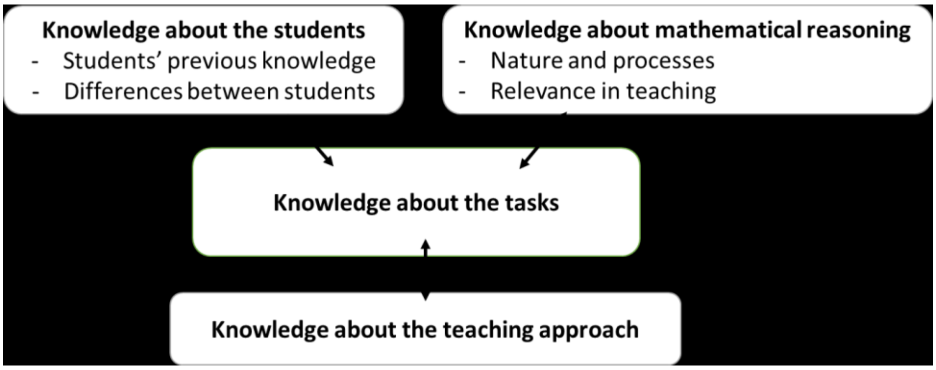

Figure 5. PTs' Knowledge about the Potential of a Task to Promote MR

The limitations of this study are related to the fact it only focuses on a couple of PTs and does not analyse the experience of other PTs who enacted their tasks in the teaching practice in different circumstances. Namely, the other PTs were required to integrate the lessons on MR in the mathematical topic that was being addressed in that period by the cooperating teacher, which could have limited their options for addressing thoroughly RM processes in the task design and would require a deeper knowledge of the curriculum. Also, the analysis does not include how the tasks were effectively enacted in the teaching practice, neither the PTs' actions in the classroom or how they reflect on their actions, which could contribute to a deeper understanding of the PT's knowledge (Ponte & Chapman, 2015) in promoting students' MR. In addition, the results cannot be seen apart from PTs' learning throughout the teacher education program, as it is not expected that a teacher develops the knowledge necessary to teach in just one course. Even so, the results of this study show valuable aspects of the type of work carried out in the instructional unit that emphasized the importance of providing PTs the opportunity to develop their knowledge to promote students' MR, by bringing them closer to the expected reality of its future practice, and through the articulation of theory with teaching practice in school.

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